$$(3) \Rightarrow \sum_{a<n\leq b} f(n) = \int_a^b f(x)\,dx + \int_a^b (\{x\})\, f'(x)\,dx$$

$$\Rightarrow f(a) - f(a) + \sum_{a<n\leq b} f(n) = \int_a^b f(x)\,dx + \int_a^b (x-[x])\, f'(x)\,dx$$

Add & sub

$$\Rightarrow \left[f(a) + \sum_{a<n\leq b} f(n)\right] - f(a) = \int_a^b f(x)\,dx + \int_a^b (x-[x]-1/2)\, f'(x)\,dx$$

$$+ \frac{1}{2}\int_a^b f'(x)\,dx$$

$$\Rightarrow \sum_{n=a}^{b} f(n) = \int_a^b f(x)\,dx + \int_a^b (x-[x]-1/2)\, f'(x)\,dx$$

$$+ \frac{1}{2}[f(x)]_a^b + f(a)$$

$$\Rightarrow \sum_{n=a}^{b} f(n) = \int_a^b f(x)\,dx + \int_a^b (x-[x]-1/2)\, f'(x)\,dx$$

$$+ \frac{1}{2}[f(b) - f(a)] + f(a)$$

$$\Rightarrow \sum_{n=a}^{b} f(n) = \int_a^b f(x)\,dx + \int_a^b (x-[x]-1/2)\, f'(x)\,dx + \frac{(f(a)+f(b))}{2}.$$

Hence proved.

Monotonically Increasing Integrators Upper and Lower Integrals

Definition:

Upper stieltjes sum and Lower stieltjes sum:

Let $P = \{x_0, x_1, x_2, \ldots x_n\}$ be a partition of $[a,b]$

And let $M_k(f) = l.u.b\, \{f(x) : x \in [x_{k-1}, x_k]\}$

and $m_k(f) = g.l.b\, \{f(x) : x \in [x_{k-1}, x_k]\}$

The numbers

$U(P, f, \alpha) = \sum_{k=1}^{n} M_k(f)\, \Delta\alpha_k$ and

$L(P, f, \alpha) = \sum_{k=1}^{n} m_k(f)\, \Delta\alpha_k$ are called respectively the upper and lower Stieltjes sum of f w.r.t α for the partition p.

where $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$

Note:

If α is increasing on $[a,b]$ then we have $\Delta\alpha_k \geq 0$ but always $m_k(f) \leq M_k(f)$

If α increasing on $[a,b]$ then we have

$$m_k(f)\,\Delta\alpha_k \leq M_k(f)\,\Delta\alpha_k$$

$$\sum_{k=1}^{n} m_k(f)\,\Delta\alpha_k \leq \sum_{k=1}^{n} M_k(f)\,\Delta\alpha_k$$

$$L(P,f,\alpha) \leq U(P,f,\alpha)$$

If $t_k \in [x_{k-1}, x_k]$, we have

$$m_k(f) \leq f(t_k) \leq M_k(f)$$

$$\sum m_k(f)\,\Delta\alpha_k \leq \sum f(t_k)\,\Delta\alpha_k \leq \sum M_k(f)\,\Delta\alpha_k$$

$$\boxed{L(P,f,\alpha) \leq S(P,f,\alpha) \leq U(P,f,\alpha)}$$

Theorem: 11

Assume that α increasing on $[a,b]$, then

(i) p' is finer than p, we have

$$U(P',f,\alpha) \leq U(P,f,\alpha) \text{ and}$$

$$L(P',f,\alpha) \geq L(P,f,\alpha)$$

(ii) For any two partitions p_1 and p_2 we have

$$L(P_1,f,\alpha) \leq U(P_2,f,\alpha)$$

Proof: Let α increasing on $[a,b]$

Let p and p' be any two partitions of $[a,b]$

It is enough if we prove the theorem,

if P contains exactly one more point than P.

Say c

Let P be the partition $\{x_0, x_1, \ldots x_k, \ldots x_n\}$

Let P' be the partition $\{x_0, x_1, \ldots x_{k-1}, c, x_k, \ldots x_n\}$

Then clearly $P' \supseteq P$

Then $U(P', f, \alpha) = M_1(f)\Delta\alpha_1 + M_2(f)\Delta\alpha_2 + \ldots + M'[\alpha(c) - \alpha(x_{k-1})]$

$+ M''[\alpha(x_k) - \alpha(c)] + M_{k+1}(f)\Delta\alpha_{k+1} + \ldots + M_n(f)\Delta\alpha_n$

$$\Rightarrow U(P', f, \alpha) = \sum_{\substack{i=1 \\ i \neq k}}^{n} M_i(f)\Delta\alpha_i + M'[\alpha(c) - \alpha(x_{k-1})] + M''[\alpha(x_k) - \alpha(c)] \quad \text{--- (1)}$$

where $M' = l.u.b\{f(x) : x \in [x_{k-1}, c]\}$

$M'' = l.u.b\{f(x) : x \in [c, x_k]\}$

$M_k(f) = l.u.b\{f(x) : x \in [x_{k-1}, x_k]\}$

then we have,

$M' \leq M_k(f)$ and

$M'' \leq M_k(f)$

Using the above conditions in (1) we have

$$U(P', f, \alpha) \leq \sum_{\substack{i=1 \\ i \neq k}}^{n} M_i(f)\Delta\alpha_i + M_k(f)[\alpha(c) - \alpha(x_{k-1})] + M_k(f)[\alpha(x_k) - \alpha(c)]$$

$$\Rightarrow U(P', f, \alpha) \leq \sum_{\substack{i=1 \\ i \neq k}}^{n} M_i(f)\Delta\alpha_i + M_k(f)[\alpha(x_k) - \alpha(x_{k-1})]$$

$$\leq \sum_{\substack{i=1 \\ i \neq k}}^{n} M_i(f)\Delta\alpha_i + M_k(f)\Delta\alpha_k$$

$$\leq \sum_{i=1}^{n} M_i(f)\Delta\alpha_i$$

$\leq U(P, f, \alpha)$

$\Rightarrow U(P', f, \alpha) \leq U(P, f, \alpha)$

Now,

To prove: $L(P', f, \alpha) \geq L(P, f, \alpha)$

Consider,

$$L(P', f, \alpha) = m_1(f)\Delta\alpha_1 + m_2(f)\Delta\alpha_2 + \cdots + m'[\alpha(c) - \alpha(x_{k-1})] + m''[\alpha(x_k) - \alpha(c)] + m_{k+1}(f)\alpha_{k+1} + \cdots + m_n(f)\Delta\alpha_n$$

$$= \sum_{\substack{i=1 \\ i\neq k}}^{n} m_i(f)\Delta\alpha_i + m'[\alpha(c) - \alpha(x_{k-1})] + m''[\alpha(x_k) - \alpha(c)] \longrightarrow (1)$$

where $m' = g.l.b \{f(x) : x \in [x_{k-1}, c]\}$

$m'' = g.l.b \{f(x) : x \in [c, x_k]\}$

Then we have

$m' \geq m_k(f)$ and $m'' \geq m_k(f)$

Using the above conditions in (1) we get

$$L(P', f, \alpha) \geq \sum_{\substack{i=1 \\ i\neq k}}^{n} m_i(f)\Delta\alpha_i + m_k(f)[\alpha(c) - \alpha(x_{k-1})] + m_k(f)[\alpha(x_k) - \alpha(c)]$$

$$= \sum_{\substack{i=1 \\ i\neq k}}^{n} m_i(f)\Delta\alpha_i + m_k(f)[\alpha(x_k) - \alpha(x_{k-1})]$$

$$= \sum_{i=1}^{n} m_i(f)\Delta\alpha_i$$

$$= L(P, f, \alpha)$$

$\Rightarrow L(P', f, \alpha) \geq L(P, f, \alpha)$

(ii) To prove that: $L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$

If P_1 and P_2 are any two partition of $[a, b]$

Let P_1, P_2 be any two partitions of $[a,b]$

Let $P = P_1 \cup P_2$

Then P is the common refinement of both P_1 and P_2

(i.e.) $P \supseteq P_1$ and $P \supseteq P_2$

Then for every $P \supseteq P_1$ we have

$$L(P_1, f, \alpha) \leq L(P, f, \alpha) \quad [\text{By (i)}]$$

and $\forall$ $P \supseteq P_2$ we have

$$U(P, f, \alpha) \leq U(P_2, f, \alpha) \quad [\text{By (i)}]$$

also we know that,

$$L(P, f, \alpha) \leq U(P, f, \alpha)$$

Then, $L(P_1, f, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P_2, f, \alpha)$

$$L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$$

for every P_1, P_2 of $[a,b]$

Hence proved.

Definition:

Assume that α is increasing on $[a,b]$, then the upper Stieltjes Integral of f w.r.t α is $\int_a^b f(x)\, d\alpha(x)$ and is denoted by $\bar{I}(f, \alpha)$ and is defined as

$$\overline{\int_a^b} f\, d\alpha = g.l.b \{U(P, f, \alpha), \forall\ P \in \mathcal{P}[a,b]\}$$

The lower stieltjes Integral of f w.r.t α on $[a,b]$ is $\int_a^b f(x)\, d\alpha(x)$ and it is denoted by $\underline{I}(f, \alpha)$ and is defined as

$$\underline{\int_a^b} f\, d\alpha = l.u.b \{L(P, f, \alpha), \forall\ P \in \mathcal{P}[a,b]\}$$

Note :

In the special case where $\alpha(x) = x$, the upper and lower sums are denoted by $U(P, f)$ and $L(P, f)$ and are called upper and lower Riemann sums

And the corresponding integrals, denoted by $\overline{\int_a^b} f\,dx$ and $\underline{\int_a^b} f\,dx$ are called upper and lower Riemann integrals.

Theorem : 12

P Assume that α increasing on $[a, b]$ then,

$\underline{I}(f, \alpha) \leq \overline{I}(f, \alpha)$

Proof: $\overline{I}(f, \alpha) = g.l.b \{U(P, f, \alpha) : P \in \mathcal{P}[a, b]\}$

Let $\epsilon > 0$ be an arbitrary

Then $\overline{I}(f, \alpha) + \epsilon$ is not the lower bound for the above criterion

$\therefore \exists$ a partition $P_1 \in \mathcal{P}[a, b]$ such that,

$U(P_1, f, \alpha) < \overline{I}(f, \alpha) + \epsilon$

But for every partition, $P \in \mathcal{P}[a, b]$ we have,

$L(P, f, \alpha) \leq U(P_1, f, \alpha)$

$\therefore$ we have,

$L(P, f, \alpha) \leq U(P_1, f, \alpha) < \overline{I}(f, \alpha) + \epsilon, \ \forall P \in \mathcal{P}[a, b]$

$\Rightarrow L(P, f, \alpha) < \overline{I}(f, \alpha) + \epsilon, \ \forall P \in \mathcal{P}[a, b]$

then $\forall P \in \mathcal{P}[a, b]$

$l.u.b \{L(P, f, \alpha) : P \in \mathcal{P}[a, b]\} < \overline{I}(f, \alpha) + \epsilon$

$\Rightarrow \underline{I}(f, \alpha) < \overline{I}(f, \alpha) + \epsilon$

Since $\epsilon > 0$ is arbitrary

$\underline{I}(f, \alpha) \leq \overline{I}(f, \alpha)$

Hence the proof.

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Example:

Let $\alpha(x) = x$ and f be a function defined on $[0,1]$ as follows.

$$f(x) = 1 \quad \text{if } x \text{ is rational}$$
$$= 0 \quad \text{if } x \text{ is irrational}$$

To show that $\underline{I}(f,\alpha) < \bar{I}(f,\alpha)$

Soln: Here, $\underline{I}(f,\alpha) = l.u.b\{L(P,f,\alpha) : P \in \mathcal{P}[a,b]\}$

and $\bar{I}(f,\alpha) = g.l.b\{U(P,f,\alpha) : P \in \mathcal{P}[a,b]\}$

where $L(P,f,\alpha) = \sum_{k=1}^{n} m_k(f)\,\Delta\alpha_k$,

$$m_k(f) = g.l.b\{f(x),\ \forall x \in [x_{k-1}, x_k] \subset [0,1]\}$$

$\Rightarrow m_k(f) = g.l.b\{0,1\} = 0$

$\Rightarrow L(P,f,\alpha) = 0$

$\Rightarrow \underline{I}(f,\alpha) = 0 \longrightarrow (1)$

where $U(P,f,\alpha) = \sum_{k=1}^{n} M_k(f)\,\Delta\alpha_k$

$$M_k(f) = l.u.b\{f(x),\ \forall x \in [x_{k-1}, x_k] \subset [0,1]\}$$

$\Rightarrow M_k(f) = l.u.b\{0,1\} = 1$

$\Rightarrow U(P,f,\alpha) = \sum_{k=1}^{n} 1\cdot[\alpha(x_{k-1}) - \alpha(x_k)]$

$$= 1\cdot\sum_{k=1}^{n}[\alpha(x_{k-1}) - \alpha(x_k)]$$

$$= 1\cdot(x_n - x_0)$$

$U(P,f,\alpha) = 1\cdot(1-0) = 1$

$\Rightarrow \bar{I}(f,\alpha) = 1 \longrightarrow (2)$

From (1) and (2)

$\underline{I}(f,\alpha) < \bar{I}(f,\alpha)$

Note :

The Same Result holds if

$f(x) = 0$, if x is rational

$f(x) = 1$, if x is irrational

Additive Linearly problem of Upper and Lower Riemann Integrals.

Upper and Lower Riemann Integrals share many of the properties of the integrals.

For eg. we have,

$$\bar{\int_a^b} f \, d\alpha = \bar{\int_a^c} f \, d\alpha + \bar{\int_c^b} f \, d\alpha \quad , \text{if } a < c < b$$ and the same equation holds for Lower Integrals.

Riemann condition:

We say that f satisfies Riemann's condition w.r.t α on $[a,b]$ if ,

$\forall \epsilon > 0$ there exists a partition P_ϵ Such that $P \supseteq P_\epsilon$ implies

$$0 \le U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

Theorem : 13

Assume that α increasing on $[a,b]$ then the following three statements are equivalent :

(i) $f \in R(\alpha)$ on $[a,b]$ iff

(ii) f satisfies Riemann's condition w.r.t α on $[a,b]$ $\Rightarrow U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$

(iii) $\underline{I}(f, \alpha) = \bar{I}(f, \alpha)$

proof:

To prove : (i) $\Rightarrow$ (ii)

Assume that statement (i) holds.

To prove that : Statement (ii) is true.

ie) Assume that $f \in R(\alpha)$ on $[a,b]$

To prove that: 'f' satisfies Riemann condition w.r.t 'α' on $[a,b]$

ie) $0 \le U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ $a = x_0 \; x_1 \; \dots \; x_n$ $\alpha(x_0) \le \alpha(x_1) \le$

Since α is increasing on $[a,b]$, $\alpha(a) \le \alpha(b)$

(whenever $a < b$)

Case : (a)

Suppose $\alpha(a) = \alpha(b)$

ie) α is constant through out $[a,b]$

then, $U(P,f,\alpha) - L(P,f,\alpha) = \sum_{k=1}^{n} M_k(f)\Delta\alpha_k - \sum_{k=1}^{n} m_k(f)\Delta\alpha_k$

$[\because \Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1}) = 0]$

$\therefore 0 \le U(P,f,\alpha) - L(P,f,\alpha) = 0 < \epsilon$ $\boxed{\epsilon > 0}$

$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

Case : (b)

Suppose that $\alpha(a) < \alpha(b)$

$|S(P,f,\alpha - A| < \epsilon$

Since $f \in R(\alpha)$ on $[a,b]$,

For any given $\epsilon > 0$, $\exists$ a partition P_ϵ of $[a,b]$ Such that for every $P \supseteq P_\epsilon$ and for every choice of $t_k, t_k' \in [x_{k-1}, x_k]$ we have. TP: $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

$$\left|S(P,f,\alpha) - \int_a^b f\,d\alpha\right| < \epsilon$$

$$\left|\sum_{k=1}^{n} f(t_k)\Delta\alpha_k - \int_a^b f\,d\alpha\right| < \frac{\epsilon}{3} \quad \text{and}$$

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$$\left|\sum_{k=1}^{n} f(t_k')\,\Delta\alpha_k - \int_a^b f\,d\alpha\right| < \frac{\epsilon}{3}$$

combining the above two inequalities we have

$$\left|\sum_{k=1}^{n}[f(t_k) - f(t_k')]\,\Delta\alpha_k\right|$$

$$= \left|\sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k - \int_a^b f\,d\alpha - \sum_{k=1}^{n} f(t_k')\,\Delta\alpha_k + \int_a^b f\,d\alpha\right|$$

$$\le \left|\sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k - \int_a^b f\,d\alpha\right| + \left|\int_a^b f\,d\alpha - \sum_{k=1}^{n} f(t_k')\,\Delta\alpha_k\right|$$

$$= \left|\sum_{k=1}^{n} f(t_k)\,\Delta\alpha_k - \int_a^b f\,d\alpha\right| + \left|\sum_{k=1}^{n} f(t_k')\,\Delta\alpha_k - \int_a^b f\,d\alpha\right|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}$$

$$\Rightarrow \left|\sum_{k=1}^{n}[f(t_k) - f(t_k')]\,\Delta\alpha_k\right| < \frac{2\epsilon}{3} \longrightarrow (1)$$

$\boxed{\because g.l.b = -l.u.b}$

For every $P \supseteq P_\epsilon$ consider

$$M_k(f) - m_k(f) = l.u.b\{f(x) : x \in [x_{k-1}, x_k]\}$$
$$- g.l.b\{f(x') : x' \in [x_{k-1}, x_k]\}$$
$$= l.u.b\{f(x) : x \in [x_{k-1}, x_k]\}$$
$$- (- l.u.b\{-f(x') : x' \in [x_{k-1}, x_k]\}$$

$[\because g.l.b\,A = -l.u.b\,(-A)]$

$$= l.u.b\{f(x) : x \in [x_{k-1}, x_k]\}$$
$$+ l.u.b\{-f(x') : x' \in [x_{k-1}, x_k]\}$$

$$M_k(f) - m_k(f) = l.u.b\{f(x) - f(x') : x, x' \in [x_{k-1}, x_k]\}$$

$[\because \operatorname{Sup} A + \operatorname{Sup} B = \operatorname{Sup}(A+B)]$

choose $h = \dfrac{\epsilon}{3[\alpha(b) - \alpha(a)]}$

Then,

$M_k(f) - m_k(f) - h$ is not an upper bound of

$\{f(x) - f(x') : x, x' \in [x_{k-1}, x_k]\}$